Celestial Holography

1/13/25 - 1/16/25 @ Sanya

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soft physics book: 1703.05448
celestial lectures: 2108.04801
survey up to '21: 2111.11392
short summary: 2310.04932

4 × 1.5 hour lectures

1) Motivation + IR Triangle Primer
2) Asymptotic Symmetries & Soft Theorems
3) Celestial Amplitudes & the Holographic Dictionary
4) Holographic Symmetry Algebras & Future Directions

Lecture 1: Motivation + IR Triangle Primer

Goals: Why flat holography?

What's different about flat spacetimes?

set up Penrose diagram for $\mathbb{R}^{1,3}$

motivate IR triangle

How have people tackled these questions?

flat lim of AdS

Carrollian CFTs

Celestial CFT

$$
\begin{tikzcd}
 & \text{Quantum Gravity} \arrow[dr, "\text{consistency in UV}"'] & \\
 & & \text{String Theory} \arrow[dl, "\text{AdS/CFT correspondence}"] \\
\text{BH thermo.} \arrow[r] & \text{Holography} &
\end{tikzcd}
$$

Holographic Principle: A theory of quantum gravity can be encoded in a lower dim theory w/out gravity at the spacetime boundary.

Celestial Holography: want to apply the holo. princ. to $\Lambda = 0$ spacetimes.

Two Approaches

Top Down: Find stringy construction.

Bottom Up: Match symmetries, identify consistency conditions.

we will follow this approach

Carrollian CFTs

dim. red. to Celestial CFT

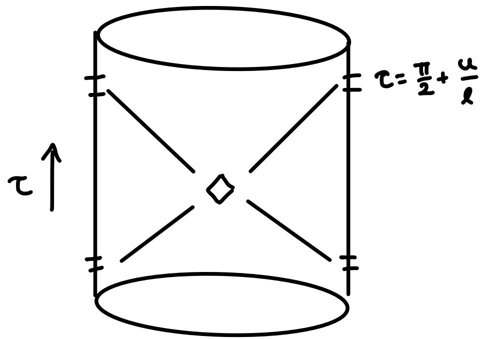
- causal structure of the boundary is different

- $\Lambda = 0$ spacetimes have an enhanced asymptotic sym. group

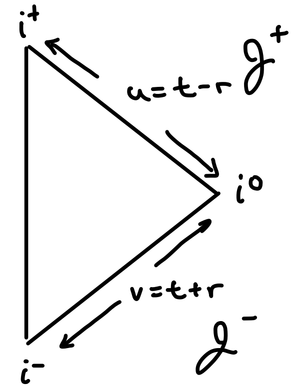
IR Triangle

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$\Lambda < 0$



$\Lambda = 0$



$$ds^2 = -\left(1 + \frac{r^2}{\ell^2}\right)dt^2 + \left(1 + \frac{r^2}{\ell^2}\right)^{-1} dr^2 + 2r^2 \gamma_{z\bar{z}}\, dz\, d\bar{z}$$

$$= -\left(1 + \frac{r^2}{\ell^2}\right) du^2 - 2\,du\,dr + 2r^2 \gamma_{z\bar{z}}\, dz\, d\bar{z}$$

flat limit $\longrightarrow -du^2 - 2\,du\,dr + 2r^2 \gamma_{z\bar{z}}\, dz\, d\bar{z}$

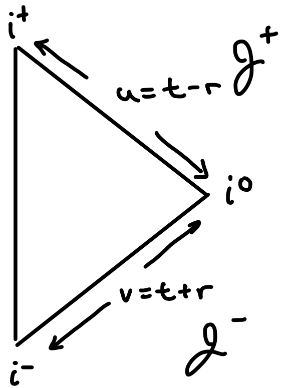
large r $\longrightarrow \sim r^2\left(-\frac{1}{\ell^2} du^2 + 2\gamma_{z\bar{z}}\, dz\, d\bar{z}\right)$

$z = e^{i\phi} \tan\frac{\theta}{2}$, $\gamma_{z\bar{z}} = \frac{2}{(1 + z\bar{z})^2}$

$t = u + \ell \arctan\frac{r}{\ell}$

$\Lambda \to 0,\ \ell \to \infty$ where $\ell^2 = \frac{3}{|\Lambda|}$

$c \sim \frac{1}{\ell}$ Carrollian limit



starting w/ null coordinates

$$u=t-r \quad , \quad v=t+r$$

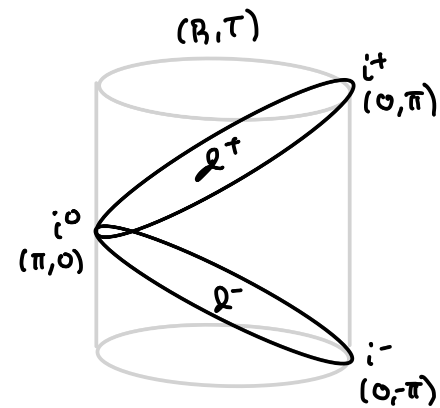
we introduce rescaled coords.

$$u=\tan U \quad , \quad v=\tan V$$

w/ $U,V \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. Then the metric in terms of

$$T=U+V \quad , \quad R=V-U$$

is conformal to a patch of $S^3 \times \mathbb{R}$



- massive particles begin at i^- and end at i^+
- massless particles begin at $\mathcal{J}^-$ and end at $\mathcal{J}^+$
- spacelike geodesics end at i^0

Asymptotically flat spacetimes

- general soln's to $G_{\mu\nu} = 8\pi G T_{\mu\nu}$

- have the same conf. boundary as $\mathbb{R}^{1,3}$ (ignoring bh's)

- have a larger asymptotic symmetry group

Bondi mass

angular mom. asp.

$$ds^2 = \overbrace{-du^2 - 2du\,dr + 2r^2\gamma_{z\bar{z}}}^{\text{flat metric}} + \frac{2m_B}{r}du^2 + \left(D^z C_{zz} + \frac{1}{r}\left[\frac{4}{3}N_z - \frac{1}{4}D_z\left(C_{zz}C^{zz}\right)\right]\right)du\,dz + c.c.$$
$$+ r C_{zz}\,dz^2 + c.c. + \dots$$

cov. deriv. on S^2

radiative dof

subleading in r

$\partial_u m_B$, $\partial_u N_z$ constrained by eom

$$\text{Asymptotic Symmetries} = \frac{\text{Allowed Symmetries}}{\text{Trivial Symmetries}}$$

Bondi, van der Burg,
Metzner, Sachs '62

look at diffeos ξ which preserve gauge
s.t. $\mathcal{L}_\xi g_{\mu\nu} \sim$ same falloffs as above

the # of ξ which act non-triv. on rad. data is ∞!

$$\begin{aligned}\xi = {} & f\partial_u - \frac{1}{r}(D^z f\partial_z + D^{\bar z} f\partial_{\bar z}) + D^z D_z f\partial_r \\ & + \left(1+\frac{u}{2r}\right)Y^z\partial_z - \frac{u}{2r}D^{\bar z}D_z Y^z\partial_{\bar z} \\ & - \frac{1}{2}(u+r)D_z Y^z\partial_r + \frac{u}{2}D_z Y^z\partial_u + \text{c.c.} + \dots\end{aligned}$$

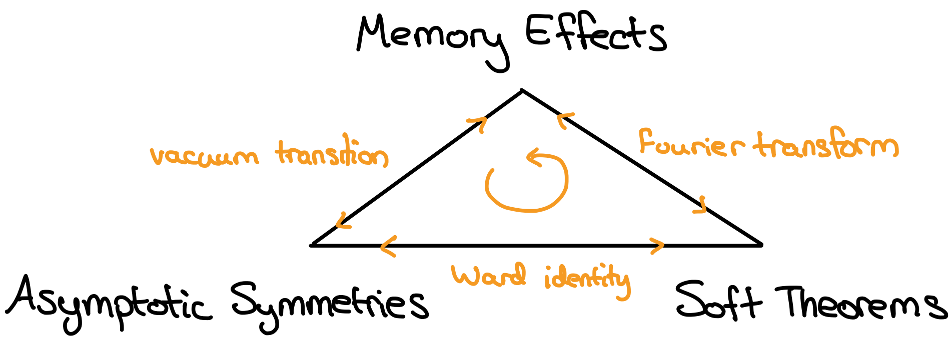
$f(z,\bar z)$ supertranslations ∞!

$Y(z)$ superrotations

vs. Poincare $\quad f = c_1 + c_2\frac{z+\bar z}{1+z\bar z} + c_3\frac{i(\bar z - z)}{1+z\bar z} + c_4\frac{1-z\bar z}{1+z\bar z}$

10! $\quad Y = a + bz + cz^2$

IR Triangle



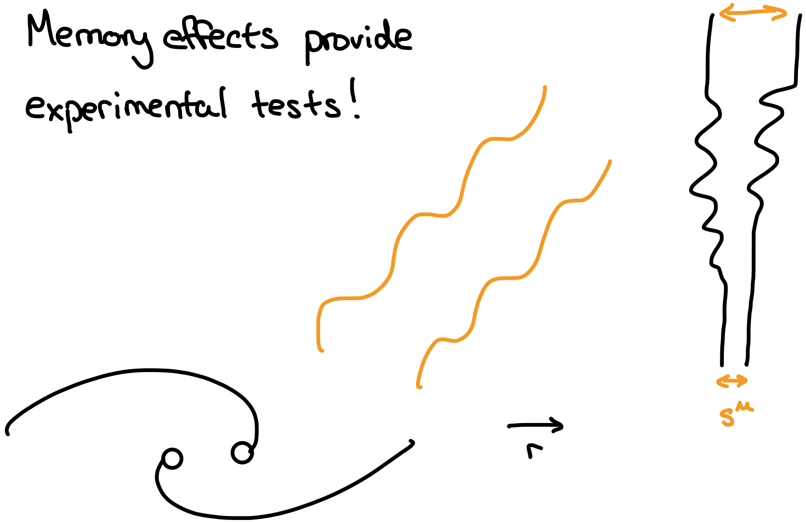
focus of next lecture

- leading E&M
- leading grav.
- subleading grav.

...

pattern → predictions

Memory effects provide experimental tests!



$$\partial_\tau^2 s^\mu = R^\mu{}_{\lambda\rho\nu} t^\lambda t^\rho s^\nu$$

geod. dev.

$$\downarrow$$

$$\partial_u^2 s^{\bar z} = \frac{\gamma^{z\bar z}}{2r} \partial_u^2 C_{zz} s^z$$

$t^\lambda \partial_\lambda \sim \partial_u$, $\tau \sim u$

$R_{zuzu} \sim -\frac{1}{2} r \partial_u^2 C_{zz}$

$$\downarrow$$

$$\Delta s^{\bar z} = \frac{\gamma^{z\bar z}}{2r} \Delta C_{zz} s^z$$

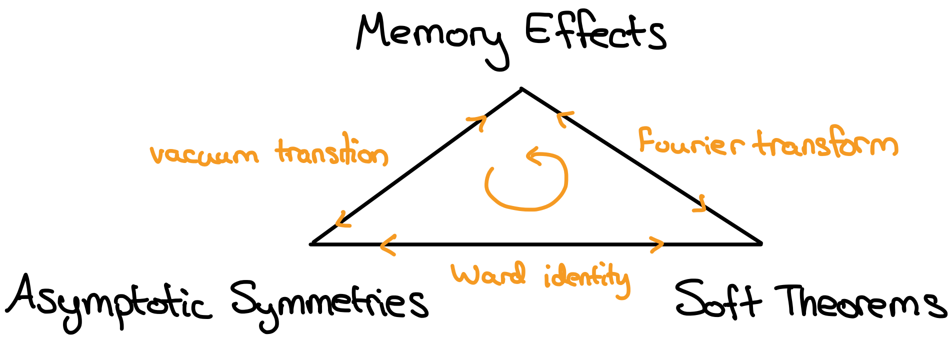
Strom. lec. ex. 13

$\exists$ non-triv. tail behavior of grav. waveform

- $\rightarrow$ meas. w/ asymp. detectors — mem. effect
- $\rightarrow$ $\Theta(u) \xrightarrow{\text{F.T.}} \frac{1}{\omega}$ — soft pole
- $\rightarrow$ $\Delta C_{zz} = -2D_z^2 \Delta C$ — vac. trans.

Lecture 2: Asymptotic Symmetries & Soft Theorems

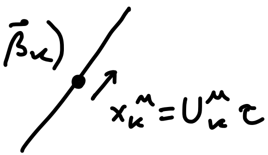
Goal: Demonstrate Asymptotic sym. $\Leftrightarrow$ soft thm. for U(1) example



focus of this lecture

Let us start by considering the electromagnetic field for a set of moving point charges with charge Q_k and 4-velocity U_k^μ

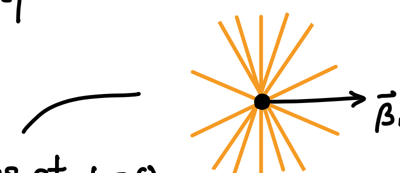
$$U_k^\mu = \gamma_k(1, \vec{\beta}_k)$$



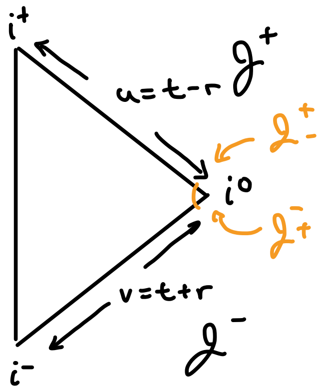
$$j^M_\mu(x) = \sum_{k=1}^{n} Q_k \int d\tau\, U_{k,\mu}\, \delta^{(4)}(x^\nu - U_k^\nu \tau)$$

The Liénard-Wiechert solution to $\nabla^\mu F_{\mu\nu} = e^2 j^M_\nu$ has

$$F_{rt}(t,\vec{x}) = \frac{e^2}{4\pi} \sum_{k=1}^{n} \frac{Q_k \gamma_k (r - t\hat{x}\cdot\vec{\beta}_k)}{|\gamma_k^2 (t - r\hat{x}\cdot\vec{\beta}_k)^2 - t^2 + r^2|^{3/2}}$$



$\vec{E}$ field lines at $t=0$



Now let's examine how things behave near the conf. bndy.

recall: $r\to\infty$, u fixed $\to \mathscr{I}^+$

$r\to\infty$, v fixed $\to \mathscr{I}^-$

also $F_{rt}=F_{ru}=F_{rv}$ due to anti-sym

$$F_{ru}\big|_{\mathscr{I}^+} = \frac{e^2}{4\pi r^2}\sum_{k=1}^{n}\frac{Q_k}{\gamma_k^2(1-\hat{x}\cdot\vec{\beta}_k)^2}$$

$$F_{rv}\big|_{\mathscr{I}^-} = \frac{e^2}{4\pi r^2}\sum_{k=1}^{n}\frac{Q_k}{\gamma_k^2(1+\hat{x}\cdot\vec{\beta}_k)^2}$$

$$\lim_{r\to\infty} r^2 F_{ru}(\hat{x})\big|_{\mathscr{I}^+_-} = \lim_{r\to\infty} r^2 F_{rv}(-\hat{x})\big|_{\mathscr{I}^-_+}$$

antipodal matching!

this will play an important role

What is the ASG for $U(1)$ gauge theory?

$$S = \frac{-1}{4e^2}\int d^4x\sqrt{-g}\,F_{\mu\nu}F^{\mu\nu} + S_M$$

$$\xRightarrow{\delta/\delta A_\mu} \quad \nabla^\mu F_{\mu\nu} = e^2 j^M_\nu \quad \text{where } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \text{ and } \nabla^\mu j^M_\mu = 0$$

Now there is also a gauge sym $\delta_\varepsilon A_\mu = \partial_\mu \varepsilon(u,r,z,\bar{z})$ we should gauge fix

$\nabla^\mu A_\mu = 0$ still allows ε s.t. $\Box\varepsilon = 0$

residual gauge dof.

Consider the asymptotic expansion

$$\mathcal{O}(u,r,z,\bar{z}) = \sum_n r^{-n}\,\mathcal{O}^{(n)}(u,z,\bar{z})$$

Then solving $\Box\mathcal{E}=0$ order-by-order gives

$$(\Box\mathcal{E})^{(n)} = 2(n-2)\partial_u\mathcal{E}^{(n-1)} + [D^2+(n-2)(n-3)]\,\mathcal{E}^{(n-2)}$$

$\mathcal{E}^{(1)}(u,z,\bar{z})$ free data

Can use this to set $A_u^{(1)}=0$. Then

$$A_u \sim \mathcal{O}\left(\frac{1}{r^2}\right) \qquad A_r \sim \mathcal{O}\left(\frac{1}{r^2}\right) \qquad A_A \sim \mathcal{O}(1)$$

This residual gauge fixing still allows for a non-zero $\varepsilon^{(0)}(z,\bar{z})$.

$$Q_\varepsilon = \frac{1}{e^2}\int_{i^0} \varepsilon(z,\bar{z}) \star F$$

generates the non-trivial $\delta_\varepsilon A_A = \partial_A \varepsilon$ respecting our b.c.'s

$$\Longrightarrow \quad \text{ASG} \ni \text{large } U(1) \text{ gauge trans.}$$

Meanwhile bc of antipodal matching of $F_{ru}|_{\mathcal{I}^+_-}$ and $F_{rv}|_{\mathcal{I}^-_+}$:

$$Q^+_\varepsilon = \frac{1}{e^2}\int_{\mathcal{I}^+_-} \varepsilon \star F = \frac{1}{e^2}\int_{\mathcal{I}^-_+} \varepsilon \star F = Q^-_\varepsilon \qquad \text{if} \quad \varepsilon(z,\bar{z})|_{\mathcal{I}^+_-} = \varepsilon(z,\bar{z})|_{\mathcal{I}^-_+}$$

our Ward id.

antipodal pts.

Can we see $\langle \text{out} | Q^+_\varepsilon S - S Q^+_\varepsilon | \text{in} \rangle = 0$ in S-matrix elements?

leading r-behavior of u-component of eom:

$$\partial_u F^{(2)}_{ru} + D^z F^{(0)}_{uz} + D^{\bar{z}} F^{(0)}_{u\bar{z}} + e^2 j^{(2)}_u = 0 \qquad \text{(assume } m=0 \text{ charges)}$$

$$\downarrow$$

$$Q^+_\varepsilon = -\frac{1}{e^2} \int_{\mathcal{I}^+} \underbrace{du\, d^2z \left(\partial_z \varepsilon F^{(0)}_{u\bar{z}} + \partial_{\bar{z}} \varepsilon F^{(0)}_{uz} \right)}_{Q_S \text{ (photon field)}} + \int_{\mathcal{I}^+} \underbrace{du\, d^2z\, \varepsilon \gamma_{z\bar{z}} j^{(2)}_u}_{Q_H \text{ (meas. charge)}}$$

ditto for Q^-_ε.

Now $\langle \text{out} | Q_S^+$ will change the state to one w/ additional photon

$\langle \text{out} | Q_H^+$ no change in particle # soft thm!

$$A_\mu(x) = e\sum_{\alpha=\pm}\int\frac{d^3q}{(2\pi)^3}\frac{1}{2\omega_q}\left[\varepsilon_\mu^{\alpha *}(\vec{q})\, a_\alpha(\vec{q})\, e^{iq\cdot x} + \varepsilon_\mu^{\alpha}(\vec{q})\, a_\alpha(\vec{q})^\dagger e^{-iq\cdot x}\right]$$

for our gauge $F^{(0)}_{uz} = \partial_u A^{(0)}_z - \cancel{\partial_z A^{(0)}_u}$ where $A^{(0)}_z = \lim_{r\to\infty} \partial_z x^\mu A_\mu(x)$

now at large r fixed u: $e^{iq\cdot x} = e^{-i\omega_q u - i\omega_q r(1-\cos\theta)} \to e^{-i\omega_q u} \times \frac{1}{\omega_q r}\frac{\delta(\theta)}{\sin\theta}$

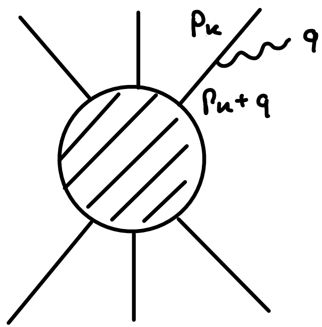
$$\Rightarrow A^{(0)}_z = \frac{-i}{8\pi^2}\frac{\sqrt{2}e}{1+z\bar{z}}\int_0^\infty d\omega_q\left[a_+(\omega_q\hat{x})e^{-i\omega_q u} - a_-(\omega_q\hat{x})^\dagger e^{i\omega_q u}\right]$$

$$\Rightarrow \int du\, F^{(0)}_{uz} = \frac{-1}{8\pi}\frac{\sqrt{2}e}{1+z\bar{z}}\lim_{\omega\to 0^+}\left[\omega a_+(\omega\hat{x}) + \omega a_-(\omega\hat{x})^\dagger\right]$$

so $\langle out| Q_S^+ S - S Q_S^- |in\rangle = -\langle out| Q_H^+ S - S Q_H^- |in\rangle$ from our Ward id

but we know this from soft thms!

Weinberg tells us these insertions have a universal form!



$$\langle out | a_+(\vec{q}) S | in \rangle = e \sum_{out-in} \frac{Q_k\, p_k \cdot \varepsilon^+}{p_k \cdot q} \langle out | S | in \rangle + \mathcal{O}(\omega_q^0)$$

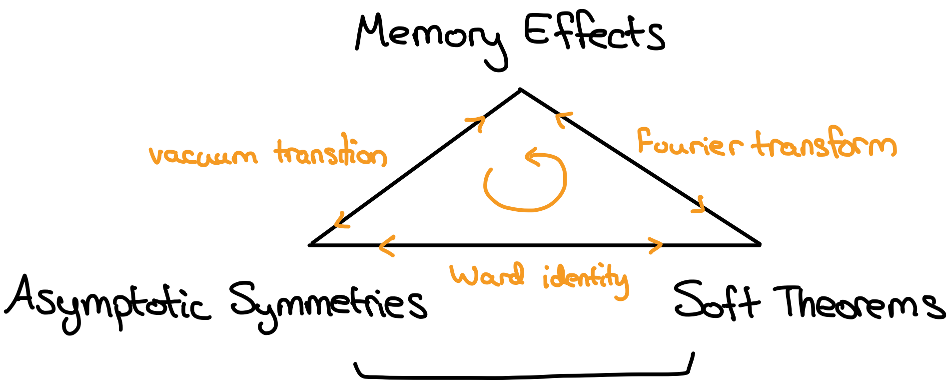
Plugging in the soft theorem

$$\langle out | \int du\, F_{uz}^{(0)} S | in \rangle = \frac{-e^2}{4\pi} \sum_{out-in} \frac{Q_k}{z - z_k} \langle out | S | in \rangle$$

we indeed have $\langle out | Q_\varepsilon^+ S - S Q_\varepsilon^- | in \rangle = 0$

Ward id $\Longleftrightarrow$ soft thm!

Looking back at the IR triangle



- leading E&M — large U(1)
- leading grav. — Supertranslations
- NEW! ⇒ subleading grav. — Super rotations
- ...

pattern → predictions

But look! $j^{+} := Q_S^{+}\left(\varepsilon = \frac{1}{z-\omega}\right) = -4\pi \int du\, F_{uz}$ obeys

$$\langle j(z)\,\mathcal{O}_1(z_1,\bar{z}_1)\ldots\mathcal{O}_n(z_n,\bar{z}_n)\rangle = \sum_k \frac{Q_k}{z-z_k}\langle \mathcal{O}_1(z_1,\bar{z}_1)\ldots\mathcal{O}_n(z_n,\bar{z}_n)\rangle$$

ASG $\Rightarrow$ 2D Kac-Moody sym. of S-matrix?

Next time: reorganize scattering to make these symmetries manifest!

CCFT!

Lecture 3: Celestial Amplitudes & the Holographic Dictionary

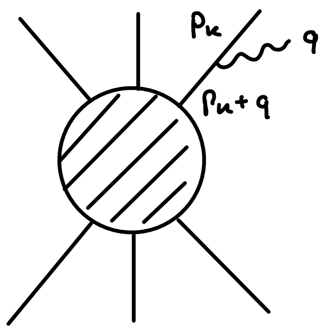
Last time: ASG Ward id = soft thm $\Rightarrow$ 2D current

leading soft photon $\Longleftrightarrow$ large $U(1)$ $\leftarrow$ saw this last time

leading soft graviton $\Longleftrightarrow$ supertranslations

subleading soft graviton $\Longleftrightarrow$ superrotations $\leftarrow$ let's explore!

While we focused on the U(1) case last time, the same construction generalizes to gravity. In this case the soft theorem is universal up to subleading order



$$\langle out | a_+(\vec{q}) S | in \rangle = \left(S^{(0)\pm} + S^{(1)\pm} \right) \langle out | S | in \rangle + \mathcal{O}(\omega)$$

$$S^{(0)\pm} = \frac{\kappa}{2} \sum_k \eta_k \frac{(p_k \cdot \varepsilon^{\pm})^2}{p_k \cdot q} \qquad S^{(1)\pm} = -i \frac{\kappa}{2} \sum_k \eta_k \frac{p_{k\mu} \varepsilon^{\pm\mu\nu} q^{\lambda} J_{k\lambda\nu}}{p_k \cdot q}$$

These can be recast as supertranslation and superrotation Ward identities, respectively.

In Lecture 1 we wrote down the AFS metric

$$ds^2 = \underbrace{-du^2 - 2du\,dr + 2r^2\gamma_{z\bar{z}}}_{\text{flat metric}} + \frac{2m_B}{r}du^2 + \left(D^z C_{zz} + \frac{1}{r}\left[\frac{4}{3}N_z - \frac{1}{4}D_z(C_{zz}C^{zz})\right]\right)du\,dz + c.c.$$
$$+ r C_{zz}\,dz^2 + c.c. + \ldots$$

Bondi mass

angular mom. asp.

cov. deriv. on S^2

radiative dof

subleading in r

$\partial_u m_B$, $\partial_u N_z$ constrained by eom

and identified the ASG

$$\xi = f\partial_u - \frac{1}{r}(D^z f\partial_z + D^{\bar{z}} f\partial_{\bar{z}}) + D^z D_z f\partial_r$$
$$+ \left(1+\frac{u}{2r}\right)Y^z\partial_z - \frac{u}{2r}D^{\bar{z}}D_z Y^z\partial_{\bar{z}}$$
$$-\frac{1}{2}(u+r)D_z Y^z\partial_r + \frac{u}{2}D_z Y^z\partial_u + c.c. + \ldots$$

$f(z,\bar{z})$ supertranslations

$Y(z)$ superrotations

∞ vs. Poincare 10!

we did not write down the charges

$$Q^{+}[f,Y] = \frac{1}{8\pi G}\int_{\mathcal{I}^{+}_{-}} [2m_B(f+\frac{u}{2}D_A Y^A) + Y^A N_A]$$

but the manipulations to a Ward id statement are analogous to the U(1) case and involve $\int du$ of the leading-in-r part of the eom

$$G_{ui} = 8\pi G T_{ui} \qquad i \in \{u, z, \bar{z}\}$$

which we can again split into a soft and hard part.

For today we will just need the following takeaway:

subsoft grav $\Rightarrow$ 2D stress tensor

or in equations:

$$\langle T_{zz} \mathcal{O}_1 \dots \mathcal{O}_n \rangle = \sum_k \left[\frac{h_k}{(z-z_k)^2} + \frac{\partial_{z_k}}{(z-z_k)} \right] \langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle$$

$h_k = \frac{1}{2}(s_k - \omega_k \partial_{\omega_k})$

diagonalized for boost eigenstates

$$T_{zz} = -i \frac{3!}{8\pi G} \int d^2 w \, \frac{1}{(z-w)^4} \int du \, u \partial_u C^{w}{}_{\bar{w}}$$

Today: Kinematics of scattering $\to$ Celestial Amplitudes

Claim: ASG sym enhancements naturally organized in terms of a CFT_2

$$SL(2,\mathbb{C}) \simeq \text{Lorentz} \subset \text{Poincaré} \qquad\qquad \text{Global Conf.}$$

$\downarrow$ + gravity

$$\text{Vir} \times \text{Vir} \simeq \text{Superrotations} \subset \text{BMS} \qquad\qquad \text{Larger sym. multiplets!}$$

$$_{\text{boost}}\langle \text{out} | S | \text{in} \rangle_{\text{boost}} = \langle \mathcal{O}^{\pm}_{\Delta_1, J_1}(z_1, \bar{z}_1) \dots \mathcal{O}^{\pm}_{\Delta_n, J_n}(z_n, \bar{z}_n) \rangle_{\text{CCFT}}$$

is just a change of basis!

Today we will focus on the global part. We can prepare scattering states that are 2D primaries with an appropriate choice of wavepacket.

Def'n: A conformal primary wavefunction is a fn on $\mathbb{R}^{1,3}$ which transf as

$$\Phi^{s}_{\Delta,J}\left(\Lambda^{\mu}{}_{\nu}X^{\nu};\frac{aw+b}{cw+d},\frac{\bar{a}\bar{w}+\bar{b}}{\bar{c}\bar{w}+\bar{d}}\right)=(cw+d)^{\Delta+J}(\bar{c}\bar{w}+\bar{d})^{\Delta-J}D_s(\Lambda)\Phi^{s}_{\Delta,J}(X^{\mu};w,\bar{w})$$

For on-shell states we impose the spin-s lin. eoms. The Lorentz inv. guarantees

$$\mathcal{O}^{s,\pm}_{\Delta,J}(w,\bar{w})=i\left(\hat{\mathcal{O}}^{s}(X),\Phi^{s}_{\Delta^*,-J}(X^{\mu}_{\mp};w,\bar{w})\right)_{\Sigma}$$

$$X^0_{\pm}=X^0\mp i\varepsilon$$

is a 2D primary operator.

It is straightforward to construct for any spin. Using

$$q^\mu = (1+w\bar{w},\ w+\bar{w},\ i(\bar{w}-w),\ 1-w\bar{w}) \qquad \varepsilon_w^\mu = \frac{1}{\sqrt{2}}\partial_w q^\mu$$

we can construct a null tetrad

$$l^\mu = \frac{q^\mu}{-q\cdot X} \qquad n^\mu = X^\mu + \frac{X^2}{2} l^\mu \qquad m^\mu = \varepsilon_w^\mu + (\varepsilon_w \cdot X)\, l^\mu$$

and we have

$$\Phi^s_{\Delta, J=+s} = m_{\mu_1} \ldots m_{\mu_s} \frac{f(X^2)}{(q\cdot X)^\Delta}$$

mass shell cond. determines $f(X^2)$

This works for any m, but for $m=0$ things are simpler!

$$A^{\pm}_{\mu;\Delta,J=+1} = m_\mu \frac{1}{(q\cdot X_\pm)^\Delta} = c(\Delta)\,\varepsilon_{\omega;\mu} \underbrace{\int_0^\infty d\omega\, \omega^{\Delta-1}}_{\text{Mellin transform}} e^{\pm i\omega q\cdot X - \varepsilon\omega} + \nabla_\mu \lambda^{\pm}_{\Delta,J}$$

As such we can just Mellin transform the states

$$|\Delta, s; 0, 0\rangle = \int_0^\infty d\omega\, \omega^{\Delta-1} |p=\omega(1,0,0,1); s\rangle$$

or amplitudes

$$\langle \mathcal{O}^{\pm}_{\Delta_1,J_1} \cdots \mathcal{O}^{\pm}_{\Delta_n,J_n} \rangle_{CCFT} = \prod_{i=1}^{n} \int_0^\infty d\omega_i\, \omega_i^{\Delta_i-1} \langle out|S|in\rangle$$

to learn about the Celestial holographic dictionary.

Let's look at $|\Delta, s; 0, 0\rangle = \int_0^\infty d\omega\, \omega^{\Delta-1} |p=\omega(1,0,0,1); s\rangle$. The combo's

$$L_0 = \tfrac{1}{2}(M^{12} + i M^{+-}), \quad L_{-1} = \tfrac{1}{2}(-M^{2+} - i M^{1+}), \quad L_1 = \tfrac{1}{2}(-M^{2-} + i M^{1-})$$
$$\bar{L}_0 = \tfrac{1}{2}(-M^{12} + i M^{+-}), \quad \bar{L}_{-1} = \tfrac{1}{2}(M^{2+} - i M^{1+}), \quad \bar{L}_1 = \tfrac{1}{2}(M^{2-} + i M^{1-})$$

$$P_{1/2,1/2} = P^+, \quad P_{-1/2,-1/2} = P^-, \quad P_{1/2,-1/2} = P^1 - i P^2, \quad P_{-1/2,1/2} = P^1 + i P^2$$

where $x^\pm = x^0 \pm x^3$ obey the Poincare alg

$n, m \ni \mathbb{Z}$
$k, l \ni \mathbb{Z} + \frac{1}{2}$
$\longrightarrow$ BMS!

$$[L_m, L_n] = (m-n) L_{m+n} \qquad [\bar{L}_m, \bar{L}_n] = (m-n) \bar{L}_{m+n}$$
$$[L_n, P_{k,l}] = \left(\tfrac{n}{2} - k\right) P_{k+n,l} \qquad [\bar{L}_n, P_{k,l}] = \left(\tfrac{n}{2} - l\right) P_{k,l+n}$$

Then no-cont. spin $\Rightarrow$ $L_1|\Delta,s\rangle = 0$, $\bar{L}_1|\Delta,s\rangle = 0$ h.w. cond

change of basis $\Rightarrow$ $L_0|\Delta,s\rangle = \frac{1}{2}(\Delta+s)|\Delta,s\rangle$, $\bar{L}_0|\Delta,s\rangle = \frac{1}{2}(\Delta-s)|\Delta,s\rangle$

while P^0-P^3, $P^1\pm iP^2$ annihilate this state and $P^0+P^3 : \Delta \mapsto \Delta+1$

$\omega\in(0,\infty) \rightarrow \Delta\in 1+i\lambda$ Principal series spectrum to capture radiative phase space

tension w/ h.w. cond. and action of translations related to distributional nature

$$\langle \Delta_1, J_1; z_1, \bar{z}_1 | \Delta_2, J_2; z_2, \bar{z}_2\rangle = \delta(\Delta_1-\Delta_2)\,\delta^{(2)}(z_1-z_2)\,\delta_{J_1J_2}$$

$L_n^{+} = -\bar{L}_n$, $P_{a,b}^{+} = P_{b,a}$ ———— exotic 2D CFT!

Now let us turn to the $m=0$ Celestial Amplitudes

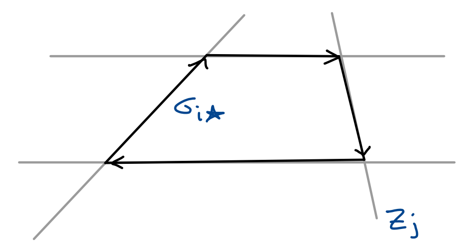
$$\langle \mathcal{O}^{\pm}_{\Delta_1,J_1} \cdots \mathcal{O}^{\pm}_{\Delta_n,J_n} \rangle_{CCFT} = \prod_{i=1}^{n} \int_0^\infty d\omega_i \, \omega_i^{\Delta_i - 1} \langle out | S | in \rangle$$

Saw 2-pt fn distributional. More generally $A(\omega_i, z_i, \bar{z}_i) = M \times \delta^{(4)}(\sum p_i)$

Letting $s = \sum \omega_i$, $\sigma_i = s^{-1} \omega_i$

$$\prod_{i=1}^{n} \int_0^\infty d\omega_i \, \omega_i^{\Delta_i - 1} (\cdot) = \int_0^\infty ds \, s^{-1+\sum \Delta_i} \prod_{i=1}^{n} \int_0^1 d\sigma_i \sigma_i^{\Delta_i - 1} \delta(\sum \sigma_i - 1)(\cdot)$$

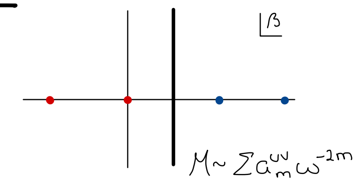
we see that for $n \leq 5$ the σ_i are localized.



Taking a closer look at $2\to 2$, where $\mathcal{M}(s,t)$ is the stripped amplitude the $m=0$ celestial amplitude takes the form

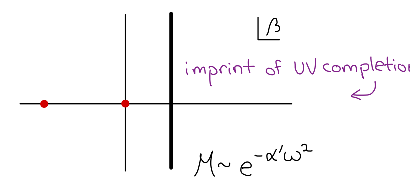
$$\tilde{A} = X\, A(\beta, z), \quad \beta = \sum \Delta_i - 1, \quad z = \frac{z_{12}\, z_{23}}{z_{13}\, z_{24}}$$

$$X = \prod_{i<j}^{4} z_{ij}^{h/3 - h_i - h_j}\, \bar{z}_{ij}^{\bar{h}/3 - \bar{h}_i - \bar{h}_j}\, \delta(i(z-\bar{z}))$$



The stripped amplitude is probed at all energy scales

$$A(\beta, z) = \int_0^\infty d\omega\, \omega^{\beta-1} \mathcal{M}(\omega^2, -z\omega^2)$$



UV behavior affects convergence and structure of poles in β.

While some of the features of the celestial Amplitudes make it look like an exotic CFT, in the rest of these lectures we will explore how far we can get treating our flat hologram as a 2D CFT.

In this paradigm the physics is encoded in the CFT data

Spectrum: $\Delta \in 1 + i\lambda$ for single part. states (what about soft limits?)

OPE data: Should describe collinear limits

want both for tomorrow's lect!

Statement 1: powers in ω turn into poles in Δ. Using $\lim_{\varepsilon\to 0}\frac{\varepsilon}{2}\omega^{\varepsilon-1}=\delta(\omega)$

$$\langle \text{out}|S|\text{in}\rangle = \omega^{-1}A^{(-1)}+A^{(0)}+\dots \Rightarrow \lim_{\Delta\to -n}(\Delta+n)\int_0^\infty d\omega\,\omega^{\Delta-1}\sum_k \omega^k A^{(k)} = A^{(n)}$$

$\exists$ poles at -ive integer Δ whose residues are terms in the soft exp.

Statement 2: Celestial OPEs can be extracted from splitting fn.

$$\lim_{z_{ij}\to 0} A_n(p_1,\dots,p_n) \to \sum_{s\in\pm 2}\text{Split}^{s}_{s_i s_j}(p_i,p_j)A_{n-1}(P=p_i+p_j)$$

$$\Downarrow$$

$$\mathcal{O}_{\Delta_1,2}(z_1,\bar{z}_1)\mathcal{O}_{\Delta_2,2}(z_2,\bar{z}_2) \sim -\frac{\kappa}{2}\frac{\bar{z}_{12}}{z_{12}}B(\Delta_1-1,\Delta_2-1)\mathcal{O}_{\Delta_1+\Delta_2,2}(z_2,\bar{z}_2)$$

Will derive this next time and use it to identify additional symmetries!

Lecture 4: Holographic Symmetry Algebras & Future Directions

Goal: Identify $w_{1+\infty}$ sym. from Celestial OPE and explore implications.

What we'll need: how to extract soft modes ✓ — last time

how to extract Celestial OPE ✓

how to extract additional currents — focus today

how to identify their algebra

At the end of last lecture we looked at how OPEs came from splitting fn.
From the momentum space amplitude we have

$$\lim_{z_{ij}\to 0} A_n(p_1,\ldots,p_n) \to \sum_{s\in\pm 2} \mathrm{Split}^{s}_{s_i s_j}(p_i,p_j)\, A_{n-1}(p_1,\ldots,P,\ldots,p_n)$$

where
$$P^\mu = p_i^\mu + p_j^\mu, \quad \omega_P = \omega_i + \omega_j$$

and the collinear splitting factors have the following non-zero components

$$\mathrm{Split}^{2}_{2\,2}(p_i,p_j) = -\frac{\kappa}{2}\frac{\bar{z}_{ij}}{z_{ij}}\frac{\omega_P^2}{\omega_i\omega_j} \qquad \mathrm{Split}^{-2}_{2\,-2}(p_i,p_j) = -\frac{\kappa}{2}\frac{\bar{z}_{ij}}{z_{ij}}\frac{\omega_j^3}{\omega_i\omega_P^2}$$

Upon performing the change of variables $\omega_i = t\omega_p$, $\omega_j = (1-t)\omega_p$

the Mellin transform hitting the splitting function takes the form

$$\int_0^\infty d\omega_i\, \omega_i^{\Delta_i - 1} \int_0^\infty d\omega_j\, \omega_j^{\Delta_j - 1}\, \mathrm{Split}^2_{22}(\cdot) = -\frac{\kappa}{2}\frac{\bar{z}_{ij}}{z_{ij}}\left[\int_0^1 dt\, t^{\Delta_i - 1}(1-t)^{\Delta_j - 2}\right]\int_0^\infty d\omega_p\, \omega_p^{\Delta_i + \Delta_j - 1}(\cdot)$$

and similarly for $\mathrm{Split}^{-2}_{2\,-2}(p_i, p_j)$, giving the OPEs

$$\mathcal{O}_{\Delta_1,2}(z_1,\bar{z}_1)\,\mathcal{O}_{\Delta_2,2}(z_2,\bar{z}_2) \sim -\frac{\kappa}{2}\frac{\bar{z}_{12}}{z_{12}} B(\Delta_1 - 1, \Delta_2 - 1)\,\mathcal{O}_{\Delta_1+\Delta_2,2}(z_2,\bar{z}_2)$$

$$\mathcal{O}_{\Delta_1,2}(z_1,\bar{z}_1)\,\mathcal{O}_{\Delta_2,-2}(z_2,\bar{z}_2) \sim -\frac{\kappa}{2}\frac{\bar{z}_{12}}{z_{12}} B(\Delta_1 - 1, \Delta_2 + 3)\,\mathcal{O}_{\Delta_1+\Delta_2,-2}(z_2,\bar{z}_2)$$

Now this OPE also closes on the residues

$$H^k = \lim_{\varepsilon \to 0} \varepsilon\, \mathcal{O}_{k+\varepsilon,2} \qquad k = 2, 1, 0, -1 \ldots$$

Let's understand these residues better. Recall from last time that if

$$A := \langle \mathrm{out} | S | \mathrm{in} \rangle \sim \omega^{-1} A^{(-1)} + A^{(0)} + \ldots$$

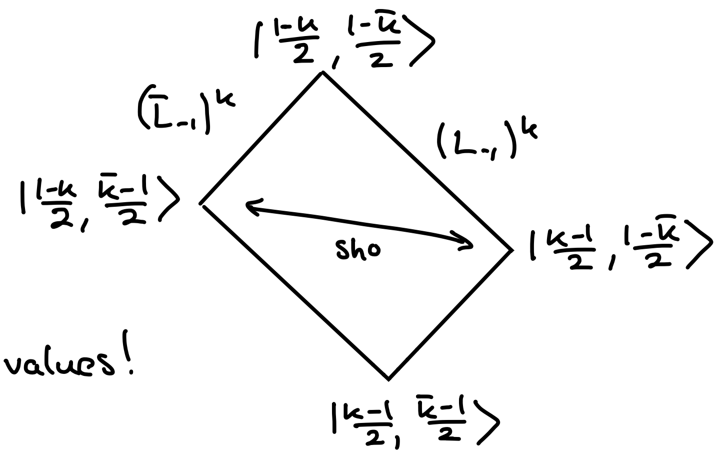
then $\int_0^\Lambda d\omega\, \omega^{\Delta-1} \omega^a = \frac{\Lambda^{\Delta+a}}{\Delta+a}$ points to poles at $\Delta = 1, 0, \ldots$ For nice UV behavior we can use $\lim_{\varepsilon \to 0} \frac{\varepsilon}{2} \omega^{\varepsilon-1} = \delta(\omega)$ to write

$$\lim_{\Delta \to -n} (\Delta + n) \int_0^\infty d\omega\, \omega^{\Delta-1} \sum_k \omega^k A^{(k)} = A^{(n)}$$

We thus see that the residues in Δ correspond to coefficients of the soft expansion in ω. From a CCFT pov these -ive int. values are special.

$$[L_1,(L_{-1})^k] = k(L_{-1})^{k-1}(2L_0+k-1) \Rightarrow L_1(L_{-1})^k|h,\bar{h}\rangle = k\underbrace{(2h+k-1)}_{=0}(L_{-1})^{k-1}|h,\bar{h}\rangle$$

$\exists$ a primary descendent when $h=\frac{1-k}{2}$ for $k \in \mathbb{Z}_>$.



* note in CCFT can tune Δ to these values!

Now $H^k = \lim_{\varepsilon \to 0} \varepsilon \mathcal{O}_{k+\varepsilon, 2}$ has weight

$$(h, \bar{h}) = \left(\frac{k+2}{2}, \frac{k-2}{2} \right) \qquad k = 2, 1, 0, -1 \ldots$$

exactly where these multiplets truncate. We can thus write

$$H^k(z, \bar{z}) = \sum_{n = \frac{k-2}{2}}^{\frac{2-k}{2}} \frac{H^k_n(z)}{\bar{z}^{\,n + \frac{k-2}{2}}}$$

and try to ask what algebra we would get from the commutator

$$[A, B](z) = \oint_z \frac{dw}{2\pi i} A(w) B(z) \leftarrow \text{from trad. radially ordered } CFT_2$$

Writing out the OPE including antiholomorphic descendents

$$\mathcal{O}_{\Delta_1,2}(z_1,\bar{z}_1)\,\mathcal{O}_{\Delta_2,2}(z_2,\bar{z}_2) \sim -\frac{\kappa}{2}\frac{1}{z_{12}}\sum_{n=0}^{\infty} B(\Delta_1-1+n,\Delta_2-1)\frac{\bar{z}_{12}^{\,n+1}}{n!}\bar{\partial}^n\mathcal{O}_{\Delta_1+\Delta_2,2}(z_2,\bar{z}_2)$$

the conformally soft modes close

$$H^k(z_1,\bar{z}_1)\,H^\ell(z_2,\bar{z}_2) \sim -\frac{\kappa}{2}\frac{1}{z_{12}}\sum_{n=0}^{1-k}\binom{2-k-\ell-n}{1-\ell}\frac{\bar{z}_{12}^{\,n+1}}{n!}\bar{\partial}^n H^{k+\ell}(z_2,\bar{z}_2)$$

and the currents obey

$$[H^k_m, H^\ell_n] = -\frac{\kappa}{2}[n(2-k)-m(2-\ell)]\frac{\left(\frac{2-k}{2}-m+\frac{2-\ell}{2}-n-1\right)!}{\left(\frac{2-k}{2}-m\right)!\left(\frac{2-\ell}{2}-n\right)!}\frac{\left(\frac{2-k}{2}+m+\frac{2-\ell}{2}+n-1\right)!}{\left(\frac{2-k}{2}+m\right)!\left(\frac{2-\ell}{2}+n\right)!}H^{k+\ell}_{m+n}$$

With a clever rescaling

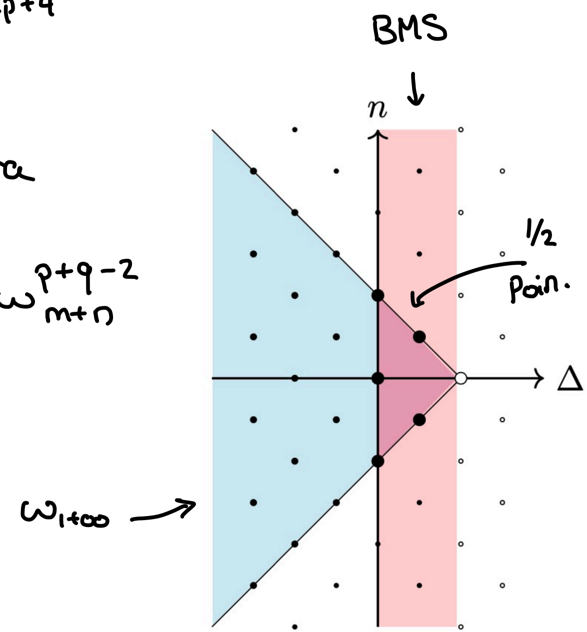
$$w_n^p := \frac{1}{\kappa}(p-n-1)!\,(p+n-1)!\,H_n^{-2p+4}$$

we can recognize this as the $\Lambda Lw_{1+\infty}$ algebra

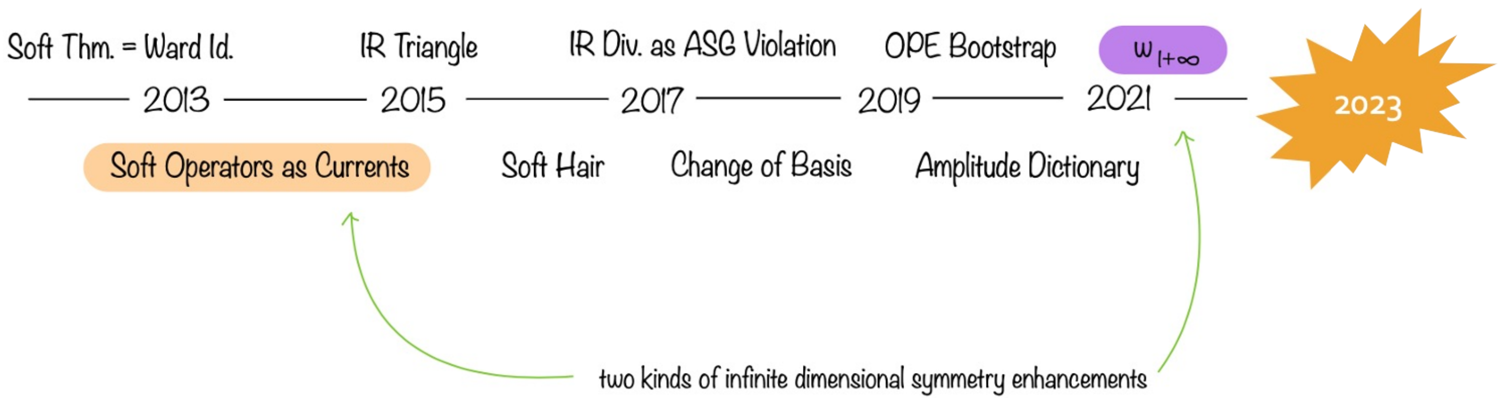
$$[w_m^p, w_n^q] = [m(q-1)-n(p-1)]\,w_{m+n}^{p+q-2}$$

Before: ASG $\rightarrow$ angle-dep ∞ sym enh. $\longrightarrow$ 2D CFT

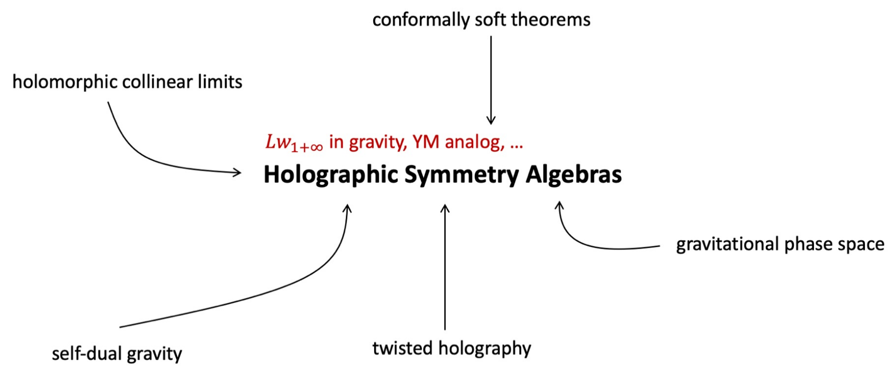
Now: coll. split. $\rightarrow$ Celestial OPE $\longrightarrow$ $w_{1+\infty}$ sym



We see that taking the 2D CFT proposal seriously leads to an even richer symmetry structure of the $\Lambda=0$ hologram.

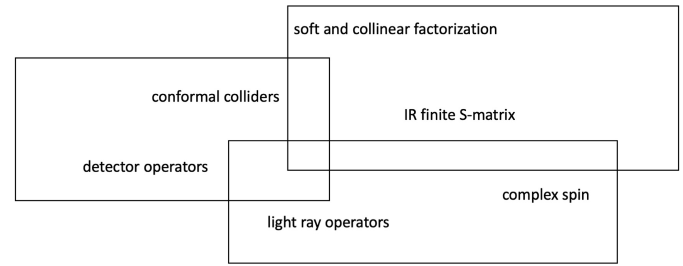


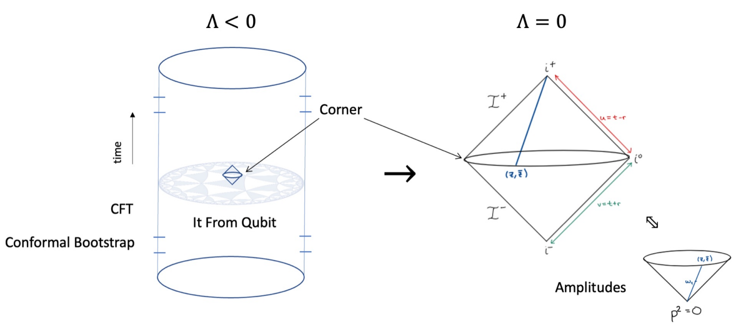
But this $w_{1+\infty}$ symmetry was familiar from twistor space!



Understanding the holo. symmetry algebras as chiral algebras in twisted holography led to the first top-down construction of a CCFT!

We hope we can leverage this collision of subfields to connect to other ventures interested in flat holography...





... and in the shorter term there are many adjacent fields studying closely related objects for different reasons!